STABILITY OF FLOW OF A STRUCTURALLY VISCOUS FLUID IN THE BOUNDARY LAYER ON A FLAT PLATE

V. B. Lemberskii

Inzhenerno-Fizicheskii Zhurnal, Vol. 12, No. 3, pp. 317–321, 1967

UDC 536.526

The method of small perturbations has been used to study the hydrodynamical stability of flow of a structurally viscous fluid in the boundary layer on a flat plate.

Following [1], we shall understand a structurally viscous fluid to be an incompressible fluid, obeying Newton's law, with a variable dynamic viscosity:

$$\tau_{ji} = \mu V_{ji} \quad (i,\ j = 1,\ 2,\ 3). \tag{1}$$

In our case the value of the viscosity is determined by the components of the rate of deformation tensor. The dynamical viscosity can be a function only of the invariants of the tensor V_{ji}. If we equate to zero the linear invariant we have the continuity equation of an incompressible fluid. For plane-parallel flow, the third invariant also is equal to zero. It is, therefore, natural to suppose that the dynamic viscosity μ depends only on the quadratic invariant of the rate of deformation tensor I, taken with the opposite sign. In conformity with [1], we shall represent μ by the series

$$\mu = 2\mu_0 + 4\mu_1 I^{1/2} + \ldots, \tag{2}$$

where

$$I = \frac{1}{8}\left(\frac{\partial w_p}{\partial x_q} + \frac{\partial w_q}{\partial x_p}\right)\left(\frac{\partial w_p}{\partial x_q} + \frac{\partial w_q}{\partial x_p}\right) \quad (p,\ q = 1,\ 2,\ 3).$$

In [1], for the case of a plane-parallel flow, the fluidity of a structurally viscous fluid φ was represented in the form of a series of powers of the shear stress τ. We see that this relation may be obtained by substituting Eq. (2) into the law of viscous friction (1), and converting the series, which immediately gives the following expressions for the coefficients:

$$\mu_0 = 1/\varphi_0,\ \mu_1 = -\Theta/\varphi_0^3,\ \ldots. \tag{3}$$

Here, of course, it is assumed that $\varphi_0 \neq 0$, i. e., a fluid with a non-zero limit of fluidity (for example, Bingham fluids) is excluded from examination. As was shown in [1], we may add a whole series of high-polymer, colloid, and coarsely dispersed fluids to the class of structurally viscous media, for which the viscosity at a given pressure and temperature is a single-valued function of the shear stress, where, in the range of shear stress of practical importance, we may limit ourselves to the first two terms of the expansion in describing the behavior of the fluid.

We shall write the equations of motion and continuity in the form

$$\rho\frac{\partial w_i}{\partial t} + \rho w_j\frac{\partial w_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}, \tag{4}$$

$$\frac{\partial w_i}{\partial x_i} = 0. \tag{5}$$

Substituting expressions (1) and (2) into Eq. (4), allowing for the continuity equation (5), we obtain

$$\rho\frac{\partial w_i}{\partial t} + \rho w_j\frac{\partial w_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + (\mu_0 + 2\mu_1 I^{1/2})\frac{\partial^2 w_i}{\partial x_j \partial x_j} + $$
$$+ \mu_1 I^{-1/2}\frac{\partial I}{\partial x_j}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) + \ldots. \tag{6}$$

We shall introduce the dimensionless variables

$$\xi = x/l,\ \omega = w/w_0,\ \vartheta = w_0 t/l,\ \pi = p/\rho w_0^2. \tag{7}$$

The equation of motion takes the form

$$\frac{\partial \omega_i}{\partial \vartheta} + \omega_j\frac{\partial \omega_i}{\partial \xi_j} = -\frac{\partial \pi}{\partial \xi_i} + \left(\frac{1}{\mathrm{Re}_0} - \frac{J^{1/2}}{\Theta_*}\right)\frac{\partial^2 \omega_i}{\partial \xi_j \partial \xi_j} - $$
$$- \frac{J^{-1/2}}{2\Theta_*}\frac{\partial J}{\partial \xi_j}\left(\frac{\partial \omega_i}{\partial \xi_j} + \frac{\partial \omega_j}{\partial \xi_i}\right) + \ldots, \tag{8}$$

where

$$J = \frac{\partial \omega_l}{\partial \xi_k}\frac{\partial \omega_l}{\partial \xi_k} + \frac{\partial \omega_k}{\partial \xi_l}\frac{\partial \omega_l}{\partial \xi_k},$$
$$\mathrm{Re} = \rho\varphi_0 l w_0,$$
$$\Theta_* = \rho\varphi_0^3 l^2/\Theta. \tag{9}$$

We shall impose a small perturbation on the basic steady flow. Assuming that the velocity and pressure of the main flow satisfy the equations of motion and continuity, we find, by neglecting terms of high order of small quantities

$$\frac{\partial \omega_i^0}{\partial \vartheta} + \bar{\omega}_j\frac{\partial \omega_i^0}{\partial \xi_j} + \omega_j^0\frac{\partial \bar{\omega}_i}{\partial \xi_j} = -\frac{\partial \pi^0}{\partial \xi_i} + $$
$$+ \left(\frac{1}{\mathrm{Re}_0} - \frac{\bar{J}^{1/2}}{\Theta_*}\right)\frac{\partial^2 \omega_i^0}{\partial \xi_j \partial \xi_j} - $$
$$- \frac{\bar{J}^{-1/2}}{2\Theta_*}\frac{\partial \bar{J}}{\partial \xi_j}\left(\frac{\partial \omega_i^0}{\partial \xi_j} + \frac{\partial \omega_j^0}{\partial \xi_i}\right) + \ldots, \tag{10}$$

$$\frac{\partial \omega_i^0}{\partial \xi_i} = 0, \tag{11}$$

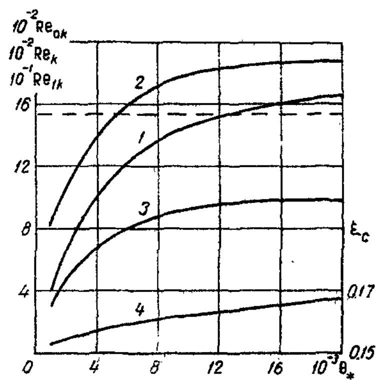
where

$$\bar{J} = \frac{\partial \bar{\omega}_l}{\partial \xi_k}\frac{\partial \bar{\omega}_l}{\partial \xi_k} + \frac{\partial \bar{\omega}_k}{\partial \xi_l}\frac{\partial \bar{\omega}_l}{\partial \xi_k}.$$

We shall make an approximate calculation of the basic flow in the boundary layer on a plane-parallel plate. It has one velocity component ω on the axis η parallel to the plate, and depends only on the coordinate ξ. We shall restrict examination to 2-dimensional perturbations, which enables us to introduce a stream function of the disturbed motion ψ. Eliminating the pressure from Eq. (10), we obtain, allowing for the above assumptions,

$$\frac{\partial \Delta\psi}{\partial \vartheta} + \omega \frac{\partial \Delta\psi}{\partial \eta} - \omega'' \frac{\partial \psi}{\partial \eta} = \left(\frac{1}{\mathrm{Re}_0} - \frac{\omega'}{\Theta_*}\right) \Delta^2\psi -$$
$$- \frac{\omega''}{\Theta_*}\left(2 \frac{\partial^3 \psi}{\partial \xi^3} - \frac{\partial h \psi}{\partial \eta}\right) - \frac{\omega'''}{\Theta_*} h\psi + \ldots . \quad (12)$$

Equation (12) is linear with respect to the stream function ψ. Its coefficient does not depend on η and ϑ.



Dependence of the numbers Re_{0k} (1), Re_k (2), Re_{1k} (3) and the coordinate ξ_c (4) on the parameter Θ_*. The dashed line is the value of the Reynolds number, corresponding to loss of stability in the flow of a Newtonian fluid in the boundary layer on a plate.

It may therefore have particular solutions of the form

$$\psi = \sigma(\xi) \exp i\alpha(\eta - c\vartheta). \quad (13)$$

Substituting expression (13) into Eq. (12), we find, following simple transformations,

$$i\alpha[(\omega - c)(\sigma'' - \alpha^2\sigma) - \omega''\sigma] =$$
$$= \left(\frac{1}{\mathrm{Re}_0} - \frac{\omega'}{\Theta_*}\right)(\sigma'''' - 2\alpha^2\sigma'' + \alpha^4\sigma) -$$
$$- \frac{1}{\Theta_*}[2\omega''\sigma''' + (\omega''' - i\alpha\omega'')(\sigma'' + \alpha^2\sigma)] + \ldots . \quad (14)$$

When $\Theta_* \to \infty$ (Newtonian fluid) Eq. (14) goes over to the well-known Orr–Sommerfeld equation.

The theory of stability of a Newtonian fluid indicates that the viscosity has a very strong influence on the perturbed motion near the point $\xi = \xi_c$, where $\omega = c$ [2]. Therefore we shall study the function σ in the immediate vicinity of this point.

We shall represent the velocity of the main flow and its derivatives in the form of series of powers of $(\xi - \xi_c)$, and restrict ourselves to the first terms of the expansions:

$$\omega - c = \omega_c'(\xi - \xi_c),$$
$$\omega^{(n)} = \omega_c^{(n)}. \quad (15)$$

We shall introduce the new variable

$$\zeta = (\xi - \xi_c)\varepsilon, \quad (16)$$

where

$$\varepsilon = (\alpha \mathrm{Re}_n)^{1/3},$$
$$\mathrm{Re}_n = \mathrm{Re}_0 \Theta_*/(\Theta_* - \omega_c' \mathrm{Re}_0).$$

Then

$$\sigma^{(n)}(\xi) = \gamma^{(n)}(\zeta)\varepsilon^n. \quad (17)$$

Comparing Eqs. (14)–(17), we obtain

$$i[\omega_c'\zeta(\varepsilon\gamma'' - \alpha^2\varepsilon^{-1}\gamma) - \omega_c''\gamma] =$$
$$= \varepsilon\gamma'''' - 2\alpha^2\varepsilon^{-1}\gamma'' + \alpha^4\varepsilon^{-3}\gamma - \left(\frac{\mathrm{Re}_n}{\mathrm{Re}_0} - 1\right)\left[\frac{2\omega_c''}{\omega_c'}\gamma''' +\right.$$
$$\left. + \frac{\omega_c''' - i\alpha\omega_c''}{\omega_c'}(\varepsilon^{-1}\gamma'' + \alpha^2\varepsilon^{-3}\gamma)\right] + \ldots . \quad (18)$$

We shall seek a solution of this equation in the form of the asymptotic expansion

$$\gamma = \gamma_0 + \gamma_1/\varepsilon + \gamma_2/\varepsilon^2 + \ldots . \quad (19)$$

Substituting the series (19) into Eq. (18), we obtain equations for γ_0, γ_1, $\gamma_2, \ldots$. Since the expression for γ_0 differs from the analogous equation in the case of flow of a Newtonian fluid only in that the Reynolds number is replaced by the number Re_n, to determine the coordinate ξ_c and the critical value of Re_{nk}, we may use the approximate estimates proposed by Lin [2], which in our notation, after a number of elementary transformations, have the form

$$\xi_c = 3c/2\omega_0' + 0.58\omega_c'^3/2\pi\omega_0'^2\omega_c'', \quad (20)$$
$$\mathrm{Re}_{nk} = 25\omega_0'/c^4. \quad (21)$$

The arguments presented above are valid for plane-parallel flows. We shall examine the specific case of motion of a structurally viscous fluid along a flat plate and shall use an expression for the velocity of the basic flow [1], which, after simple transformations, may be written in the form

$$\omega = \mathrm{Re}_0 \frac{c_f}{2}\left(\xi - \xi^3 + \frac{\xi^4}{2}\right) + \frac{\mathrm{Re}_0^3}{\Theta_*}\frac{c_f^2}{4} \times$$
$$\times \left(\xi - 2\xi^3 + \xi^4 + \frac{9}{5}\xi^5 - 2\xi^6 + \frac{4}{7}\xi^7\right), \quad (22)$$

where

$$\frac{c_f}{2} = -\frac{35\Theta_*}{26\mathrm{Re}_0^2} + \sqrt{\left(\frac{35\Theta_*}{26\mathrm{Re}_0^2}\right)^2 + \frac{35\Theta_*}{13\mathrm{Re}_0^3}}.$$

The system (20)–(22) was solved by Newton's method on an electronic M-20 computer at the Computer Center, Siberian Division, AS USSR. Calculations were also made of the critical values of the numbers Re_k and

$$\mathrm{Re}_k = \rho\varphi_c l w_0, \quad (23)$$
$$\mathrm{Re}_{1k} = \mathrm{Re}_k \xi_c^2 \omega_c'. \quad (24)$$

Results of the calculations are shown in the figure. It may be seen that the nature of the variation of the numbers Re_{0k}, Re_k, Re_{1k} as a function of the parameter Θ_* is the same.

Intensification of the non-Newtonian properties (reduction of the parameter Θ_*) lowers the stability of the flow of a structurally viscous fluid in the boundary layer on a plate.

NOTATION

τ_{ji} denotes the components of the stress deviator; μ is dynamic viscosity; V_{ji} denotes the components of the rate of deformation deviator; I is the quadratic invariant of the rate of deformation tensor, taken with the reverse sign; w_i denotes the components of the velocity vector; ξ_i denotes the coordinates of a rectangular system of coordinates; φ_0 is the fluidity for zero shear; Θ is the coefficient of structural instability; ρ is density; t is time; w_0 is the characteristic velocity (in the case of longitudinal flow over a flat plate, w_0 is the velocity of the external flow); l is the characteristic dimension (in the case of longitudinal flow over a flat plate, l is the boundary layer thickness); $\overline{\omega}_i$ is the dimensionless component of the velocity vector of the main flow; $\overline{\pi}$ is the dimensionless pressure of the main flow; ω_i^0 is the dimensionless component of the velocity vector of the disturbed flow; π^0 is the fluctuating component of the dimensionless pressure; η is the coordinate axis parallel to the plate; ξ is the coordinate axis perpendicular to the plate; ω is the velocity of the main flow in the boundary layer; ψ is the stream function of the disturbed motion; σ is the amplitude of a small perturbation; α is the wave number; c is a complex quantity characterizing the rate of propagation of the oscillation and the degree of its amplification (or damping); ξ_c is the dimensionless coordinate of the point at which $\omega = c$; $\omega_c^{(n)}$ is the derivative of the velocity of the main flow of order n at the point $\xi = \xi_c$; ω_0' is the first derivative of the main flow at $\xi = 0$; Re_n is the reduced Reynolds number; φ_c is the true fluidity at the point $\xi = \xi_c$; $h = (\partial^2/\partial\xi^2 - \partial^2/\partial\eta^2)$. The subscript k corresponds to the point of loss of stability.

REFERENCES

1. S. S. Kutateladze, V. I. Popov, and E. M. Khabakhpasheva, ZhPMTF [Journal of Applied Mechanics and Technical Physics], no. 1, 1966.

2. C. C. Lin, Quart. Appl. Math., **3**, nos. 2–4, 1945.

22 September 1966

Institute of Thermal Physics Siberian Division AS USSR, Novosibirsk